

Matrix Variate Logistic Regression Model with Application to EEG Data

HUNG HUNG*

Institute of Epidemiology and Preventive Medicine, National Taiwan

University, Taipei, Taiwan

hhung@ntu.edu.tw

CHEN-CHIEN WANG

Institute of Statistical Science, Academia Sinica, Taipei, Taiwan

Abstract

Logistic regression has been widely applied in the field of biomedical research for a long time. In some applications, covariates of interest have a natural structure, such as being a matrix, at the time of collection. The rows and columns of the covariate matrix then have certain physical meanings, and they must contain useful information regarding the response. If we simply stack the covariate matrix as a vector and fit the conventional logistic regression model, relevant information can be lost, and the problem of inefficiency will arise. Motivated from these reasons, we propose in this paper the matrix variate logistic (MV-logistic) regression model. Advantages of MV-logistic regression model include the preservation of the inherent matrix structure of covariates and the parsimony of parameters needed. In the EEG Database Data Set, we successfully extract the structural effects of covariate

*To whom correspondence should be addressed.

matrix, and a high classification accuracy is achieved.

KEY WORDS: Asymptotic theory; Logistic regression; Matrix covariate; Regularization; Tensor objects.

1 Introduction

Logistic regression has been widely applied in the field of biomedical research for a long time. It aims to model the logit transformation of conditional probability of an event as a linear combination of covariates. After obtaining the data, maximum likelihood estimates (MLE) can then be used to conduct subsequent statistical analysis such as prediction and interpretation. When the number of covariates exceeds the available sample size as is usually the case in modern research, the penalized logistic regression (Lee and Silvapulle, 1988; Le Cessie and Van Houwelingen, 1992) is proposed to estimate parameters by maximizing the penalized log-likelihood function, and to avoid the problems due to high dimensionality. Recently, penalized logistic regression has been applied to the field of biomedical research, such as cancer classification, risk factor selection, and gene interaction detection, etc. See Zhu and Hastie (2004) and Park and Hastie (2008) among others. We refer the readers to Muculloch and others (2008) and Hastie and others (2009) for details and further extensions of logistic regression.

In some applications, covariates of interest have a natural matrix structure at the time of collection. For example, research interest focuses on predicting the disease status of a patient based on his/her image of certain organ. In this situation, we actually obtain the data set of the form $\{(Y_i, X_i)\}_{i=1}^n$ which are random copies of (Y, X) , where Y is a binary random variable with value 1 indicating diseased and 0 otherwise, and X is a $p \times q$ covariate matrix representing the corresponding image of the i -th patient. Another example, the Electroencephalography (EEG) Database Data Set which will be analyzed in this article later, concerns the relationship between the genetic predisposition and alcoholism. In this study, $Y = 1$ and $Y = 0$ represent alcoholic and control groups, respectively, and the covariate X is a 256×64 matrix, where its (i, j) -th element $X_{(i,j)}$ is the measurement of voltage value at time point i and channel of electrode j .

With matrix covariate, we usually stack X column by column as a pq -vector, say $\text{vec}(X)$, and subsequent statistical analysis proceeds in the usual way. Specifically, conventional logistic regression model admits that each $X_{(i,j)}$ possesses its own effect ξ_{ij} on the response Y . Motivated from the matrix structure of X , it is natural to store ξ_{ij} 's into a $p \times q$ parameter matrix ξ . Then, conventional logistic regression model takes the formulation

$$\begin{aligned} \text{logit}\{P(Y = 1|X)\} &= \gamma + \sum_{i,j} \xi_{ij} X_{(i,j)} \\ &= \gamma + \text{vec}(\xi)^T \text{vec}(X), \end{aligned} \quad (1)$$

where γ is the intercept term. It is obvious that model (1) does not consider the inherent matrix structure of X , as the corresponding parameter matrix ξ enters model (1) only through $\text{vec}(\xi)$ while its matrix structure is ignored. As X is a matrix at the time of collection, it is reasonable for ξ to possess certain structure. For example, those $X_{(i,j)}$'s in a common region or in the same column (row) of X may have similar effects on the response. Directly fitting (1) without considering the structural relationships among ξ_{ij} 's will cause the problem of inefficiency. Moreover, by ignoring the inherent matrix structure, one can hardly identify the row and column effects from their joint effect matrix ξ , when the structural effects of X are the interest of the study. Another problem of model (1) is that the number of parameters usually becomes extremely large in comparison with the sample size. For instance, we have $1 + 64 \times 256 = 16385$ parameters for the EEG Database Data Set when fitting model (1), while the available sample size is 122 only. As high dimensionality makes statistical inference procedure unstable, it becomes urgent to seek a more efficient method in dealing with matrix covariate. The main theme of this research is thus to overcome the above mentioned problems, via incorporating the hidden structural information of X into statistical modeling.

The rest of this article is organized as follows. Section 2 introduces the matrix variate logistic regression model. Its statistical meaning is also discussed. Section 3 deals with the asymptotic properties of our estimators and the implementation algorithm. The proposed method is further evaluated through two simulation studies in Section 4 and the EEG Database Data Set in Section 5. Section 6 illustrates the extension of matrix variate logistic regression model to the case of multiple classes. The paper is ended with

conclusions in Section 7.

2 Matrix Variate Logistic Regression Model

2.1 Model specification

To incorporate the structural information into modeling, as motivated from the matrix structure of X , we propose the matrix variate logistic (MV-logistic) regression model

$$\text{logit}\{P(Y = 1|X)\} = \gamma + \alpha^T X \beta, \quad (2)$$

where $\alpha = (\alpha_1, \dots, \alpha_p)^T$ and $\beta = (\beta_1, \dots, \beta_q)^T$ are the row and column parameter vectors, respectively, and γ is the intercept term. As the matrix structure of X is preserved in model (2), these parameters have their own physical meanings. In the EEG Database Data Set, for instance, α is interpreted as the effect of different time points, and β as the effect of different channels. By fitting MV-logistic regression model, we are able to extract the column (row) information of X which will provide further insights into the relationship between Y and X . Note that under model (2), (α, β) can only be identified up to a scale, since $(c^{-1}\alpha, c\beta)$ will result in the same model for any constant $c \neq 0$. For the sake of identifiability, without loss of generality, we assume $\alpha_1 = 1$ in our derivation (see Remark 2.1 for further discussion). Denote the rest of parameters in α by α^* , i.e., $\alpha = (1, \alpha^{*T})^T$, and $\theta = (\gamma, \alpha^{*T}, \beta^T)^T$ are the parameters of interest. We thus have $p + q$ free parameters contained in θ , while it is $pq + 1$ in the conventional logistic regression model (1). One can see that a merit of model (2) is the parsimony of parameters used. Thus, when model (2) is correctly specified, an efficiency gain is expected.

Adoption of model (2) is equivalent to modeling the covariate-specific odds ratio R_{ij} of $X_{(i,j)}$ while keeping the rest covariates fixed as

$$\log(R_{ij}) = \alpha_i \beta_j \Leftrightarrow R_{ij} = \{\exp(\beta_j)\}^{\alpha_i}. \quad (3)$$

Thus, the effect of $X_{(i,j)}$ depends on the product $\alpha_i \beta_j$ instead of α_i or β_j solely. A positive $\alpha_i \beta_j$ implies $R_{ij} > 1$. Note that since we set $\alpha_1 = 1$, if $\beta_j > 0$ (< 0), then $\alpha_i > 1$ (< 1) indicates $R_{ij} > R_{1j}$. Take the EEG Database Data Set to exemplify again, relation (3)

implies that each channel has its own baseline odds ratio $\exp(\beta_j)$. Depending on the time point of being measured, say i , the odds ratio is further modified by taking a power of α_i .

Remark 2.1. *Although at the population level there is no difference for which α_i is set to one, it is crucial to practical implementation. If α_i is near zero, setting $\alpha_i = 1$ will lead to unstable results. Here we provide an easy guidance to select the baseline effect. Let ρ_{ij} be the sample correlation coefficient between $X_{(i,j)}$ and Y . We then set $\alpha_i = 1$ if $i = \operatorname{argmax}_k \{\sum_{j=1}^q |\rho_{kj}| : k = 1, \dots, p\}$. The intuition is to choose the one as the baseline which is the most likely to be correlated with the response. We find in our numerical studies this simple approach performs well.*

Remark 2.2. *Standardization of covariate will not affect the final result of logistic regression model except a change of scale in parameter estimates, since the standard deviation of each covariate can be absorbed into the corresponding parameter. In MV-logistic regression model, however, we have pq covariates but only $p+q$ free parameters and, hence, it is generally impossible to absorb those standard deviations into fewer parameters. In summary, standardization of covariates will result in a different MV-logistic regression model. We will further discuss this issue in Section 5.*

2.2 Statistical meaning of MV-logistic regression model

This subsection devotes to investigating the statistical meaning of MV-logistic regression model. We will assume the validity of the general model (1) with (γ_0, ξ_0) being the true value of (γ, ξ) . One will see that fitting MV-logistic regression model actually aims to estimate *the best rank-1 approximation* of the true parameter matrix ξ_0 , in the sense of minimum Kullback-Leibler divergence (KL-divergence), or equivalently, maximum likelihood. This observation supports the applicability of MV-logistic regression model in practice.

First observe that, as $\alpha^T X \beta = \operatorname{vec}(\alpha \beta^T)^T \operatorname{vec}(X)$, MV-logistic regression model (2) is equivalent to the conventional model (1) with the constraint $\xi = \alpha \beta^T$. Thus, MV-logistic regression utilizes the matrix structure of ξ and approximates it by a rank-1 matrix $\alpha \beta^T$ in model fitting. Secondly, it is known that maximizing the likelihood function is equivalent to minimizing the KL-divergence (Bickel and Doksum, 2001). Let $f(y|X; \gamma, \xi)$ be the

conditional distribution function of Y given X under model (1). The KL-divergence between (γ_0, ξ_0) and any (γ, ξ) is defined as

$$KL(\gamma_0, \xi_0 \| \gamma, \xi) = E_X \left\{ \int \left(\log \frac{f(y|X; \gamma_0, \xi_0)}{f(y|X; \gamma, \xi)} \right) f(y|X; \gamma_0, \xi_0) dy \right\}, \quad (4)$$

where $E_X(\cdot)$ takes expectation with respect to X . Combining the above two facts, at the population level, fitting MV-logistic regression to estimate (γ, α, β) is equivalent to searching the minimizer of the minimization problem

$$\min_{\gamma, \alpha, \beta} KL(\gamma_0, \xi_0 \| \gamma, \alpha\beta^T). \quad (5)$$

Let (α_0, β_0) be the minimizer of (α, β) in (5). MV-logistic regression then aims to search the matrix $\alpha_0\beta_0^T$, which is called the best rank-1 approximation of the true parameter matrix ξ_0 . Notice that the optimality between ξ_0 and $\alpha_0\beta_0^T$ here is not measured by the usual Frobenius norm, but the KL-divergence in (5) instead (see Remark 2.3 for more explications). Therefore, when $\xi_0 = \alpha_0\beta_0^T$, MV-logistic regression must be more efficient than the conventional approach to estimating ξ_0 by the usual MLE argument. Even if ξ_0 does not equal $\alpha_0\beta_0^T$, we are still in favor of MV-logistic regression, especially when the sample size is relatively small. In fact, there is a trade-off between “correctness of model specification” and “efficiency of estimation”. With limited sample size available, instead of estimating the full parameter matrix ξ_0 which will suffer from the problem of unstable estimation, MV-logistic regression model aims to more efficiently estimate the best rank-1 approximation of ξ_0 .

Clearly, the performance of MV-logistic regression model relies on “how well the true parameter matrix ξ_0 can be approximated by a rank-1 matrix $\alpha_0\beta_0^T$ ”. As demonstrated in Section 5 that MV-logistic regression outperforms the conventional approach in the EEG Database Data Set, we believe this condition for ξ_0 is not restrictive based on the following two reasons. First, it is reasonable to assume that most covariates have effect sizes near zero in modern biomedical research and, hence, ξ_0 is plausible to be a low-rank matrix. The second reason is based on the characteristic of the underlying study. In the EEG study, for instance, measurements at the same time points (channels) would have similar effects. These facts reflect the potential of ξ_0 to be well explained by low rank approximation and, hence, the validity of MV-logistic regression model. The robustness of MV-logistic regression model will be further studied in Section 4.2.

Remark 2.3. Given the matrix ξ_0 , the weighted rank-1 approximation problem of ξ_0 with the weight matrix W (Lu and others, 1997; Manton and others, 2003) is to find the minimizer (α_0, β_0) of the minimization problem $\min_{\alpha, \beta} \|\xi_0 - \alpha\beta^T\|_W^2$, where $\|\cdot\|_W^2 = \text{vec}(\cdot)^T W \text{vec}(\cdot)$. The resulting matrix $\alpha_0\beta_0^T$ is called the best rank-1 approximation of ξ_0 in the sense of minimum $\|\cdot\|_W$ norm. When $W = I_{pq}$, the $\|\cdot\|_W$ norm reduces to the Frobenius norm, and $\alpha_0\beta_0^T$ can be obtained by singular value decomposition (SVD) of ξ_0 . Paralleling to this idea, fitting MV-logistic regression model (with (5) being the estimation criterion) can be treated as finding the best rank-1 approximation of ξ_0 in the sense of minimum KL-divergence (4).

3 Statistical Inference Procedure

Some notations are defined here for ease of reference. For any function g , $g^{(k)}$ represents the k -th derivative of g with respect to its argument. The parameters of interest are $\theta = (\gamma, \alpha^{*T}, \beta^T)^T$. Denote $P(Y_i = 1|X_i)$ by $\pi_i = \pi(\theta|X_i) = \exp(\gamma + \alpha^T X_i \beta) \{1 + \exp(\gamma + \alpha^T X_i \beta)\}^{-1}$. Define $\mathbf{\Pi}(\theta) = (\pi_1, \dots, \pi_n)^T$ and $\mathbf{V}(\theta) = \text{diag}(v_1, \dots, v_n)$, where $v_i = \pi_i(1 - \pi_i)$ is the conditional variance of Y_i given X_i . Let $\mathbf{Y} = (Y_1, \dots, Y_n)^T$ and $\mathbf{X}(\theta) = [\mathbf{X}_1(\theta), \dots, \mathbf{X}_n(\theta)]^T$ be an $n \times pq$ matrix, where $\mathbf{X}_i(\theta) = (1, \beta^T X_i^T C, \alpha^T X_i)^T$, $C = \partial\alpha/\partial\alpha^* = [\mathbf{0}_{p-1}, \mathbf{I}_{p-1}]^T$, $\mathbf{0}_a$ is the a -vector of zeros, and \mathbf{I}_a is the $a \times a$ identity matrix. One can treat $\mathbf{X}_i(\theta)$ as the working covariates of the i -th subject, which will be used in the development of our method. Note that C depends on which element of α is set to one. As $\alpha_1 = 1$ throughout the paper, the notation C is used for simplicity.

3.1 Estimation and Implementation

The estimation of θ mainly relies on maximum likelihood method. Given the data set $\{(Y_i, X_i)\}_{i=1}^n$, the log-likelihood function of θ is derived to be

$$\ell(\theta) = \sum_{i=1}^n Y_i(\gamma + \alpha^T X_i \beta) - \log\{1 + \exp(\gamma + \alpha^T X_i \beta)\}, \quad (6)$$

and θ can be estimated by the maximizer of $\ell(\theta)$.

In modern research of biostatistics, however, an important issue is the number of covariates could be large in comparison with the sample size. This will make the estimation

procedure unstable, or traditional methodologies may even fail. As mentioned in the previous section, the number of free parameters in model (2) is $p + q$, while it is $pq + 1$ in the conventional logistic regression model. MV-logistic regression then suffers less severity from high dimensionality. However, this can not entirely avoid the problem of instability. As we have seen in the EEG Database Data Set, the covariate X is a 256×64 matrix (i.e., 320 parameters in MV-logistic regression model), while there are only 122 observations. To overcome the difficulty of high dimensionality, Le Cessie and Van Houwelingen (1992) proposed the penalized logistic regression method. This motivates us to further consider the penalized MV-logistic regression. Let $J(\theta) \geq 0$ be a twice continuously differentiable penalty function of θ , and $\lambda \geq 0$ be the regularization parameter. We propose to estimate θ by

$$\hat{\theta}_\lambda = \underset{\theta}{\operatorname{argmax}} \ell_\lambda(\theta), \quad (7)$$

where $\ell_\lambda(\theta) = \ell(\theta) - \lambda J(\theta)$ is the penalized log likelihood. There are many choices of $J(\cdot)$ depending on different research purposes, wherein $J(\theta) = \|\theta\|^2/2$ and $J(\theta) = (\|\alpha^*\|^2 + \|\beta\|^2)/2$ are the most widely applied ones. The difference between them is whether to put the penalty on the intercept term γ or not. The regularization parameter λ should also be determined in practice. A commonly used approach is to select λ through maximizing the cross-validated classification accuracy. Other selection criteria can be found in Le Cessie and Van Houwelingen (1992).

Interpretations of the elements of $\hat{\theta}_\lambda$ have been introduced in Section 2.1. Let $\hat{\theta}_\lambda = (\hat{\gamma}, \hat{\alpha}^{*T}, \hat{\beta}^T)^T$ and $\hat{\alpha} = (1, \hat{\alpha}^{*T})^T$. The odds ratio R_{ij} in (3) is estimated by $\exp(\hat{\alpha}_i \hat{\beta}_j)$. We would also be interested in the success probability $\pi(\theta|x)$ for any given $p \times q$ matrix x , which can be estimated by $\pi(\hat{\theta}_\lambda|x)$. A discrimination rule is then to classify a subject with matrix covariate x to the “diseased” group if $\pi(\hat{\theta}_\lambda|x) > 0.5$ and the “non-diseased” group otherwise. Detailed inference procedures about θ and $\pi(\theta|x)$ will be discussed in the next subsection.

We close this subsection by introducing the iterative Newton method to obtain $\hat{\theta}_\lambda$. The gradient of $\ell_\lambda(\theta)$ (with respect to θ) is calculated to be

$$\ell_\lambda^{(1)}(\theta) = \ell^{(1)}(\theta) - \lambda J^{(1)}(\theta), \quad (8)$$

where $\ell^{(1)}(\theta) = \mathbf{X}(\theta)^T \{\mathbf{Y} - \mathbf{\Pi}(\theta)\}$. Moreover, the Hessian matrix of $\ell_\lambda(\theta)$ is derived to be

$$\ell_\lambda^{(2)}(\theta) = -H_\lambda(\theta) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sum_{i=1}^n C^T X_i (Y_i - \pi_i) \\ 0 & \sum_{i=1}^n X_i^T C (Y_i - \pi_i) & 0 \end{bmatrix}, \quad (9)$$

where

$$H_\lambda(\theta) = \mathbf{X}(\theta)^T \mathbf{V}(\theta) \mathbf{X}(\theta) + \lambda J^{(2)}(\theta). \quad (10)$$

As suggested by Green (1984), we will ignore the last term of (9) since its expectation is zero. Finally, $\hat{\theta}_\lambda$ can be obtained through iterating

$$\theta_{(k+1)} = \theta_{(k)} + \{H_\lambda(\theta_{(k)})\}^{-1} \ell_\lambda^{(1)}(\theta_{(k)}), \quad k = 0, 1, 2, \dots, \quad (11)$$

until there is no significant difference between $\theta_{(k+1)}$ and $\theta_{(k)}$, and outputs $\hat{\theta}_\lambda = \theta_{(k+1)}$. A zero initial $\theta_{(0)}$ is suggested and performs well in our numerical studies.

3.2 Asymptotic Properties

Asymptotic properties of $\hat{\theta}_\lambda$ can be derived through usual arguments of MLE. The result is summarized in Theorem 3.1 below.

Theorem 3.1. *Assume the validity of model (2) and the regularity conditions of the likelihood function. Assume also the information matrix $I(\theta) = E\{\mathbf{X}_i(\theta) v_i \mathbf{X}_i(\theta)^T\}$ is nonsingular. Then, for any fixed λ , $\sqrt{n}(\hat{\theta}_\lambda - \theta)$ converges weakly to $N(0, \Sigma(\theta))$, where $\Sigma(\theta) = \{I(\theta)\}^{-1}$.*

From Theorem 3.1, $\hat{\theta}_\lambda$ is shown to be a consistent estimator of θ . It also enables us to construct a confidence region of θ , provided we have an estimate of $\Sigma(\theta)$. This can be done by the usual empirical estimator with the unknown θ being replaced by $\hat{\theta}_\lambda$. In particular, define

$$\hat{\Sigma}(\theta) = \left(\frac{1}{n} H_\lambda(\theta)\right)^{-1} \left(\frac{1}{n} \mathbf{X}(\theta)^T \mathbf{V}(\theta) \mathbf{X}(\theta)\right) \left(\frac{1}{n} H_\lambda(\theta)\right)^{-1}. \quad (12)$$

We propose to estimate the asymptotic covariance matrix $\Sigma(\theta)$ by $\hat{\Sigma}(\hat{\theta}_\lambda)$. For any $0 < a < 1$, an approximate $100(1 - a)\%$ confidence interval of θ_i , the i -th element of θ , is

constructed to be

$$\left(\hat{\theta}_{\lambda,i} - z_{\frac{a}{2}} \frac{[\hat{\Sigma}(\hat{\theta}_{\lambda})]_i}{\sqrt{n}}, \hat{\theta}_{\lambda,i} + z_{\frac{a}{2}} \frac{[\hat{\Sigma}(\hat{\theta}_{\lambda})]_i}{\sqrt{n}} \right), \quad (13)$$

where $z_{a/2}$ is the $1-a/2$ quantile of standard normal and $[\hat{\Sigma}(\hat{\theta}_{\lambda})]_i$ denotes the i -th diagonal element of $\hat{\Sigma}(\hat{\theta}_{\lambda})$. As to making inference about $\pi(\theta|x)$ for any $p \times q$ matrix x , by applying delta method and the result of Theorem 3.1, we have

$$\sqrt{n} \left\{ \log \left(\frac{\pi(\hat{\theta}_{\lambda}|x)}{1 - \pi(\hat{\theta}_{\lambda}|x)} \right) - \log \left(\frac{\pi(\theta|x)}{1 - \pi(\theta|x)} \right) \right\} \xrightarrow{d} N(0, \sigma_{\pi}^2(\theta|x)), \quad (14)$$

where $\sigma_{\pi}^2(\theta|x) = \mathbf{x}(\theta)^T \Sigma(\theta) \mathbf{x}(\theta)$ and $\mathbf{x}(\theta) = (1, \beta^T x^T C, \alpha^T x)^T$. The asymptotic variance $\sigma_{\pi}^2(\theta|x)$ can be estimated by $\hat{\sigma}_{\pi}^2(\hat{\theta}_{\lambda}|x)$, where $\hat{\sigma}_{\pi}^2(\theta|x) = \mathbf{x}(\theta)^T \hat{\Sigma}(\theta) \mathbf{x}(\theta)$. An approximate $100(1-a)\%$ confidence interval of $\pi(\theta|x)$ is then constructed to be

$$\left(\frac{\exp(\hat{\gamma} + \hat{\alpha}^T x \hat{\beta} - z_{\frac{a}{2}} \frac{\hat{\sigma}_{\pi}(\hat{\theta}_{\lambda}|x)}{\sqrt{n}})}{1 + \exp(\hat{\gamma} + \hat{\alpha}^T x \hat{\beta} - z_{\frac{a}{2}} \frac{\hat{\sigma}_{\pi}(\hat{\theta}_{\lambda}|x)}{\sqrt{n}})}, \frac{\exp(\hat{\gamma} + \hat{\alpha}^T x \hat{\beta} + z_{\frac{a}{2}} \frac{\hat{\sigma}_{\pi}(\hat{\theta}_{\lambda}|x)}{\sqrt{n}})}{1 + \exp(\hat{\gamma} + \hat{\alpha}^T x \hat{\beta} + z_{\frac{a}{2}} \frac{\hat{\sigma}_{\pi}(\hat{\theta}_{\lambda}|x)}{\sqrt{n}})} \right) \quad (15)$$

which is guaranteed to be a subinterval of $[0, 1]$.

4 Simulation Studies

The proposed method is evaluated through simulation under two different settings. Let the parameters be set as $\gamma = 1$, $\alpha = (1, 0.5, -0.5 \mathbf{1}_{p-2}^T)^T$, and $\beta = (1, 0.5, 1, -\mathbf{1}_{q-3}^T)^T$, where $\mathbf{1}_a$ is the a -vector of ones, and the penalty function $J(\theta) = (\|\alpha^*\|^2 + \|\beta\|^2)/2$ is considered. The tuning parameters for different logistic regression models are determined from an independent simulation by maximizing the classification accuracy. Simulation results are reported with 1000 replicates.

4.1 Simulation under MV-logistic regression model

The first simulation study evaluates the proposed method under model (2) with $(p, q) = (12, 10)$. We first generate X such that $\text{vec}(X)$ follows a pq -variate normal distribution with mean zero and covariance matrix \mathbf{I}_{pq} . Conditional on X , Y is generated from model (2) with the specified θ . The averages of $\hat{\theta}_{\lambda}$ and standard errors from the diagonal elements of $\hat{\Sigma}(\hat{\theta}_{\lambda})$ in (12) are provided in Table 1 for $n = 150, 300$. For the case of small

sample size $n = 150$, biases for $\hat{\theta}_\lambda$ are detected. The biases are mainly due to the penalty term $\lambda J(\theta)$, as is the case of ridge regression. In fact, from the proof of Theorem 3.1, the bias term is derived to be $\lambda n^{-1} \{I(\theta)J^{(1)}(\theta)\}$ which is of order n^{-1} . The biases, however, are all relatively small in comparison with the corresponding standard deviations of $\hat{\theta}_\lambda$, and decrease as the sample size increases. Moreover, all the standard deviations are well estimated by the diagonal elements of $\hat{\Sigma}(\hat{\theta}_\lambda)$. These observations validate Theorem 3.1 and the proposed empirical variance estimator.

As $\hat{\gamma} + \hat{\alpha}^T X \hat{\beta} = \hat{\gamma} + (\hat{\beta}^T \otimes \hat{\alpha}^T) \text{vec}(X)$ is the critical component used in prediction, we also report the averages of similarities $u^T v / (\|u\| \|v\|)$ with $u = (\gamma, \beta^T \otimes \alpha^T)^T$ and $v = (\hat{\gamma}, \hat{\beta}^T \otimes \hat{\alpha}^T)^T$ in the last row of Table 1. Although biases arise especially for the case of $n = 150$, the similarities are not affected and have values very close to one in both cases. This means MV-logistic regression would have good performance in classification, since it is the direction of $(\gamma, \beta^T \otimes \alpha^T)^T$ that is relevant to classification. To demonstrate this, in each simulation replicate we also generate another independent data set, and calculate the classification accuracy by using $\hat{\gamma} + \hat{\alpha}^T X \hat{\beta}$. For comparison, we also fit the conventional logistic regression model (1) to obtain estimates of (γ, ξ) as if we ignore the relationship $\xi = \alpha \beta^T$, and denote it by $(\tilde{\gamma}, \tilde{\xi})$. The classification accuracy obtained from $\tilde{\gamma} + \text{vec}(\tilde{\xi})^T \text{vec}(X)$ is also calculated. The averaged classification accuracies and the winning proportions (the proportion of MV-logistic regression with higher classification accuracy over 1000 replicates) under $n = 150$ are placed in Table 2 (the row indexed by $\sigma = 0$). It is detected that MV-logistic regression produces more accurate results, and the superiority of MV-logistic regression becomes more obvious for larger (p, q) values.

4.2 Simulation violating MV-logistic regression model

In the previous numerical study, MV-logistic regression outperforms conventional logistic regression when data is generated from model (2). It is our purpose here to evaluate the performance of MV-logistic regression, when the underlying distribution departs from model (2). The same setting in Section 4.1 is used, except for each simulation replicate, Y is generated from the conventional logistic regression model (1) with $\gamma = 1$ and $\xi = \alpha \beta^T + \delta$, where each element of the $p \times q$ matrix δ is also randomly drawn from a normal distribution with mean zero and variance σ^2 . With an extra term δ , model (2) is

violated, and the magnitude of violation is controlled by σ . Both models (1) and (2) are fitted to obtain the estimates $(\tilde{\gamma}, \tilde{\xi})$ and $(\hat{\gamma}, \hat{\xi})$ with $\hat{\xi} = \hat{\alpha}\hat{\beta}^T$, respectively. We compare the classification accuracies of the predictors $\hat{\gamma} + \hat{\alpha}^T X \hat{\beta}$ and $\tilde{\gamma} + \text{vec}(\tilde{\xi})^T \text{vec}(X)$ applied on another independent data set, under different combinations of (p, q) and $\sigma = 0.1, 0.3, 0.5$. Note that $\sigma = 0$ means MV-logistic regression is the true model. To see how these σ values affect the deviation from MV-logistic regression model, we also report the corresponding explained proportions ρ_σ defined below. Let $k = p \wedge q$ and $s_1 > \dots > s_k > 0$ be the k non-zero singular values of each generated ξ . Then ρ_σ is the average (over 1000 replicates) of $s_1 / (\sum_{\ell=1}^k s_\ell)$. Here $0 < \rho_\sigma < 1$ measures how well ξ can be explained by its best rank-1 approximation. Small value of ρ_σ indicates severe violation of MV-logistic regression model.

The analysis results with $n = 150$ are placed in Table 2. For every σ , conventional logistic regression is the correct model, and its classification accuracies are irrelevant to σ but will decay rapidly as pq increases. On the other hand, MV-logistic regression is less affected by (p, q) than conventional approach as it involves only $p + q$ parameters. The cost for its parsimony of parameters is that, for any fixed (p, q) , its performance becomes worse for larger σ . Overall, for moderate deviations $\sigma \leq 0.3$, MV-logistic regression outperforms the conventional approach for every combination of (p, q) . For $\sigma = 0.5$, MV-logistic regression has lower classification accuracy at $(p, q) = (12, 10)$. We note that in this case, $\rho_\sigma = 33\%$ implies a large deviation from MV-logistic regression model, while conventional logistic regression is expected to have better performance as the number of parameters is $121 < n = 150$. For larger (p, q) values, however, MV-logistic regression will be the winner even in the most extreme case of $(p, q) = (20, 30)$, where the explained proportion ρ_σ is 22% only. It indicates that when p and q are large, the gain in efficiency (from fitting MV-logistic regression model) more easily exceeds the loss due to model misspecification. These observations show that MV-logistic regression model has certain robustness against the violation of model specification, especially for large number of covariates.

5 The EEG Database Data Set

In this section, we analyze the EEG Database Data Set to demonstrate the usefulness of MV-logistic regression. The data set consists of 122 subjects, wherein 77 of them belong to the group of alcoholism ($Y_i = 1$) and the rest 45 subjects are in the control group ($Y_i = 0$). Each subject completed a total of 120 trials under three different conditions (single stimulus, two matched stimuli, and two unmatched stimuli). In each trial, measurements from 64 electrodes placed on subject's scalp at 256 time points are collected, which results in a 256×64 covariate matrix. It is interested to distinguish two types of subjects based on the collected matrix covariates. The data set can be downloaded from the web site of UCI Machine Learning Repository (<http://archive.ics.uci.edu/ml/datasets/EEG+Database>).

The EEG Database Data Set was recently analyzed by Li and others (2010) whose main purpose focused on dimension reduction. Here we adopt a similar strategy for data preprocessing. In particular, we consider partial data set of single-stimulus experimenters only, and the averaged matrix covariates over different trials of the same subject (denoted by X_i^*) will be considered in our analysis. This data setting is same with the one used in Li and others (2010). Note that with 256×64 covariate matrix, we have 320 free parameters which is still an excess of the sample size 122. Before fitting MV-logistic regression model, the generalized low rank approximations (GLRAM) of Ye (2005) is performed to reduce the dimensionality of X_i^* first. GLRAM is an extension of principal component analysis to matrix objects, which aims to find orthogonal bases $A \in \mathbb{R}^{p \times p_0}$ and $B \in \mathbb{R}^{q \times q_0}$ with $p_0 < p$ and $q_0 < q$ such that X_i^* is well explained by the lower dimensional transformation $A^T X_i^* B$. Detailed analysis procedure is listed below.

1. Apply GLRAM to find A and B under $(p_0, q_0) = (15, 15)$. Define $\hat{X}_i^* = A^T X_i^* B$.
2. Standardize each element of \hat{X}_i^* to obtain the covariate matrix X_i .
3. Fit MV-logistic regression with $\{(Y_i, X_i)\}_{i=1}^{122}$. We apply the rule suggested in Remark 2.1 to set $\alpha_3 = 1$ and denote the rest α_i 's by α^* .

To estimate θ in Step 3, we adopt the penalty function $J(\theta) = \|\theta\|^2/2$. The penalty $\lambda = 24$ is chosen so that the leave-one-out classification accuracy is maximized. The resulting estimates of α and β are provided in Figure 1 (a)-(b) with the corresponding

95% confidence intervals constructed from (13). As many estimates of α_i 's and β_j 's are significantly different from zero, channels and measurement times surely play important roles in distinguishing alcoholic and control groups. Observe that all the estimates of α^* are smaller than 1. Thus, for those channels with $\hat{\beta}_j > 0$, the effects of measurement times are all smaller than the third time point. In other words, most channels achieve the largest odds ratio at the third time point. For the rest channels with $\hat{\beta}_j < 0$, the largest effect happens at the 14-th time point. Moreover, among all combinations of time points and channels, $X_{(3,8)}$ has the largest odds ratio (since $\hat{\alpha}_3\hat{\beta}_8$ is the largest among all $\hat{\alpha}_i\hat{\beta}_j$) and would be critical in classifying alcoholic and control groups. Figure 1 (c) provides the predicted probability of being alcoholism for every subject (by using the rest 121 subjects) as well as the 95% confidence interval from (15), and Figure 1 (d) gives the kernel density estimates of $\hat{\alpha}^T X_i \hat{\beta}$ for two types of subjects. An obvious separation of two groups is detected which demonstrates the usefulness of MV-logistic regression in classification.

The choice of $(p_0, q_0) = (15, 15)$ in Step 1 is the same with the data preprocessing step of Li and others (2010). Under this choice, we correctly classify 105 of 122 subjects (through leave-one-out classification procedure) by fitting MV-logistic regression, while the best result of Li and others (2010) from dimension folding (a dimension reduction technique that preserves the matrix structure of covariates) followed by quadratic discriminant analysis gives 97. We believe the reasons of a better performance for MV-logistic regression are twofold. First, we adopt a different data preprocessing technique GLRAM in Step 1, where Li and others (2010) use a version of (2D)²PCA (Zhang and Zhou, 2005). Hung and others (2011) show that GLRAM is asymptotically more efficient than (2D)²PCA in extracting bases, and hence it is reasonable for GLRAM to produce a better result. Second, standardization in Step 2 makes the EEG Database Data Set more suitable to fit MV-logistic regression model. Without standardization, MV-logistic regression cannot produce such a high classification accuracy. This also reflects that standardization is an important issue before fitting MV-logistic regression model. We remind the readers again that standardization of covariates will result in a different MV-logistic regression model.

For comparison, the p_0q_0 extracted covariates in Step 1 are also fitted with the con-

ventional (penalized) logistic regression of Le Cessie and Van Houwelingen (1992), where the tuning parameter is selected in the same way such that the leave-one-out classification accuracy is maximized. Table 3 provides the analysis results of both methods under different choices of (p_0, q_0) . One can see that MV-logistic regression uniformly outperforms conventional logistic regression. Moreover, the classification accuracy of the conventional approach decays rapidly as the numbers of p_0 and q_0 increase, while those of MV-logistic regression roughly remain constant. As mentioned previously, MV-logistic regression requires fewer parameters in model fitting, possesses certain robustness against model violation and, hence, an efficiency gain is reasonably expected.

Remark 5.1. *As suggested by one referee, we also compare our approach to the widely used procedure, principal component analysis (PCA) followed by logistic regression. In particular, PCA is applied to the vectorized covariates $\text{vec}(X_i^*)$. The leading r principal components are then used to fit the conventional (penalized) logistic regression model, where the tuning parameter is also selected to maximize the leave-one-out classification accuracy (see Table 4 for the results). It can be seen this widely applied approach can not produce classification accuracy higher than 0.820, while the best result of MV-logistic regression is 0.861. This reveals the limitation of the conventional $\text{vec}(X)$ -based approach which usually produces a large number of parameters.*

6 Extension to Multi-Class Response

We illustrate the extension of MV-logistic regression to the case of H classes with $H \geq 2$. Let $\tilde{Y}_i \in \{1, \dots, H\}$ be the random variable indicating which class the i -th subject belongs to. We can equivalently code \tilde{Y}_i as $Y_i = (Y_{1i}, \dots, Y_{H-1,i})^T$, where $Y_{hi} = 1$ indicates the i -th subject belongs to category h , $h = 1, \dots, H-1$, and $\{Y_{hi} = 0, h = 1, \dots, H-1\}$ means the i -th subject belongs to category H . Consider the model

$$\log \left\{ \frac{P(Y_i = h|X)}{P(Y_i = H|X)} \right\} = \gamma_h + \alpha_h^T X \beta_h, \quad h = 1, \dots, H-1, \quad (16)$$

with $\alpha_h = (1, \alpha_h^{*T})^T$ for the sake of identifiability as before. The parameter of interest is $\boldsymbol{\theta} = (\theta_1^T, \dots, \theta_{H-1}^T)^T$ with $\theta_h = (\gamma_h, \alpha_h^{*T}, \beta_h^T)^T$. This gives the covariate-specific probability $P(Y_i = h|X_i)$ to be $\pi_{hi} = \pi_h(\boldsymbol{\theta}|X_i) = \exp(\gamma_h + \alpha_h^T X_i \beta_h) \{1 + \sum_{h=1}^{H-1} \exp(\gamma_h + \alpha_h^T X_i \beta_h)\}^{-1}$

for $h = 1, \dots, H-1$, and $\pi_{Hi} = \pi_H(\boldsymbol{\theta}|X_i) = \{1 + \sum_{h=1}^{H-1} \exp(\gamma_h + \alpha_h^T X_i \beta_h)\}^{-1}$ for $h = H$. Based on the data $\{(Y_i, X_i)\}_{i=1}^n$, the log-likelihood function of $\boldsymbol{\theta}$ is derived to be

$$\ell^*(\boldsymbol{\theta}) = \sum_{i=1}^n \left[\sum_{h=1}^{H-1} Y_{hi}(\gamma_h + \alpha_h^T X_i \beta_h) - \log\{1 + \sum_{h=1}^{H-1} \exp(\gamma_h + \alpha_h^T X_i \beta_h)\} \right]. \quad (17)$$

Then, $\boldsymbol{\theta}$ is proposed to be estimated by $\hat{\boldsymbol{\theta}}_\lambda = \arg \max_{\boldsymbol{\theta}} \ell_\lambda^*(\boldsymbol{\theta})$, where $\ell_\lambda^*(\boldsymbol{\theta}) = \ell^*(\boldsymbol{\theta}) - \lambda J(\boldsymbol{\theta})$.

To apply the Newton method to obtain $\hat{\boldsymbol{\theta}}_\lambda$, we need to calculate the gradient vector and Hessian matrix of $\ell_\lambda^*(\boldsymbol{\theta})$. Let $\mathbf{X}^*(\boldsymbol{\theta}) = \text{diag}\{\mathbf{X}(\theta_1), \dots, \mathbf{X}(\theta_{H-1})\}$, where $\mathbf{X}(\cdot)$ is defined in the beginning of Section 3. Let also $\mathbf{Y}_h = (Y_{h1}, \dots, Y_{hn})^T$, $\boldsymbol{\Pi}_h(\boldsymbol{\theta}) = (\pi_{h1}, \dots, \pi_{hn})^T$ for $h = 1, \dots, H-1$, $\mathbf{Y}^* = (\mathbf{Y}_1^T, \dots, \mathbf{Y}_{H-1}^T)^T$, and $\boldsymbol{\Pi}^*(\boldsymbol{\theta}) = \{\boldsymbol{\Pi}_1(\boldsymbol{\theta})^T, \dots, \boldsymbol{\Pi}_{H-1}(\boldsymbol{\theta})^T\}^T$. Then, the gradient is derived to be $\ell_\lambda^{*(1)}(\boldsymbol{\theta}) = \mathbf{X}^*(\boldsymbol{\theta})^T \{\mathbf{Y}^* - \boldsymbol{\Pi}^*(\boldsymbol{\theta})\} - \lambda J^{(1)}(\boldsymbol{\theta})$, and the Hessian matrix (after ignoring the zero expectation term as in (10)) is given by $H_\lambda^*(\boldsymbol{\theta}) = \mathbf{X}^*(\boldsymbol{\theta})^T \mathbf{V}^*(\boldsymbol{\theta}) \mathbf{X}^*(\boldsymbol{\theta}) + \lambda J^{(2)}(\boldsymbol{\theta})$, where $\mathbf{V}^*(\boldsymbol{\theta}) = [\mathbf{V}_{ij}(\boldsymbol{\theta})]$, $\mathbf{V}_{hh} = \text{diag}\{\pi_{h1}(1 - \pi_{h1}), \dots, \pi_{hn}(1 - \pi_{hn})\}$, $h = 1, \dots, H-1$, and $\mathbf{V}_{hk} = \mathbf{V}_{kh} = \text{diag}(-\pi_{h1}\pi_{k1}, \dots, -\pi_{hn}\pi_{kn})$, $1 \leq h \neq k \leq H-1$. Finally, $\hat{\boldsymbol{\theta}}_\lambda$ is obtained by replacing (8) and (10) with $\ell_\lambda^{*(1)}(\boldsymbol{\theta})$ and $H_\lambda^*(\boldsymbol{\theta})$ in the iteration (11).

By a similar argument of Theorem 3.1, we can also deduce that $\sqrt{n}(\hat{\boldsymbol{\theta}}_\lambda - \boldsymbol{\theta})$ converges weakly to a normal distribution with mean zero and covariance matrix $\Sigma^*(\boldsymbol{\theta}) = \{I^*(\boldsymbol{\theta})\}^{-1}$, where $I^*(\boldsymbol{\theta}) = E[n^{-1} \mathbf{X}^*(\boldsymbol{\theta})^T \mathbf{V}^*(\boldsymbol{\theta}) \mathbf{X}^*(\boldsymbol{\theta})]$. Moreover, $\Sigma^*(\boldsymbol{\theta})$ can be estimated by $\hat{\Sigma}^*(\hat{\boldsymbol{\theta}}_\lambda)$, where

$$\hat{\Sigma}^*(\boldsymbol{\theta}) = \left(\frac{1}{n} H_\lambda^*(\boldsymbol{\theta}) \right)^{-1} \left(\frac{1}{n} \mathbf{X}^*(\boldsymbol{\theta})^T \mathbf{V}^*(\boldsymbol{\theta}) \mathbf{X}^*(\boldsymbol{\theta}) \right) \left(\frac{1}{n} H_\lambda^*(\boldsymbol{\theta}) \right)^{-1}. \quad (18)$$

Inference procedures are the same with what we have already established in the previous sections.

7 Conclusions

Besides the EEG example, tensor objects are frequently encountered in many applications, such as mammography images, ultrasound images, and magnetic resonance imaging (MRI) images, which are examples of order-two tensors. One can also imagine a subject with p covariates measured on q time points under t different treatments is naturally stored as an order-three tensor. In this paper we propose MV-logistic regression model, when the

covariates have a natural matrix structure (order-two tensor). Its performance is validated through the EEG Database Data Set, where in the best case of $(p_0, q_0) = (15, 15)$ we successfully classify 105 of 122 subjects to the right group. The superiority of MV-logistic regression comes from the fact that it aims to estimate the best rank-1 approximation of the true parameter matrix and the parsimony of parameters used. It is also found in our simulation studies that MV-logistic regression has certain robustness against the violation of model specification. Thus, MV-logistic regression model can be used as a good “working” model, especially when the sample size is relatively small in comparison with the number of covariates. Although we focus on matrix covariate in this article, the proposed method can be straightforwardly extended to tensor objects of higher order.

As discussed in Section 2.2, MV-logistic regression model aims to approximate the true parameter matrix in the sense of minimum KL-divergence. Here the discussion of MV-logistic regression model is only focused on the case of rank-1 approximation due to its simplicity in implementation and theoretical development, and we find this simple rank-1 model is suitable for the EEG Database Data Set. Of course we may encounter situations when rank-1 approximation is not adequate. It is therefore natural to extend this rank-1 structure to a more general rank- r setting. In particular, for any positive integer $r \leq p \wedge q$, consider the model

$$\text{logit}\{P(Y = 1|X)\} = \gamma + \text{vec}(AB^T)^T \text{vec}(X), \quad (19)$$

where $A \in \mathbb{R}^{p \times r}$ and $B \in \mathbb{R}^{q \times r}$ are the row and column parameter matrices, respectively. Note that model (19) can be treated as the conventional logistic regression model (1) with the constraint $\xi = AB^T$, and will reduce to our MV-logistic regression model (2) when $r = 1$. For the general rank- r setting, however, some issues need to be further studied, such as the problem of identifiability of parameters in (A, B) and the corresponding asymptotic properties. Moreover, the algorithm developed in this paper can not be directly applied, and also need to be revised to adapt to model (19). We believe model (19) has its wide applicability in practice, and is of great interest to investigate in a future study.

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Table 1: Averages of $\hat{\theta}_\lambda$ (Mean), averages of diagonal elements of $\hat{\Sigma}(\hat{\theta}_\lambda)$ (SE), and standard deviations of $\hat{\theta}_\lambda$ (SD) under model (2) with $(p, q) = (12, 10)$. The last row gives averages (standard deviations) of similarities (SIM) between $(\gamma, \beta^T \otimes \alpha^T)^T$ and $(\hat{\gamma}, \hat{\beta}^T \otimes \hat{\alpha}^T)^T$

	True	$n = 150$			$n = 300$		
		Mean	SD	SE	Mean	SD	SE
γ	1.000	1.095	0.447	0.460	1.055	0.289	0.287
α^*	0.500	0.549	0.165	0.157	0.525	0.098	0.096
	-0.500	-0.543	0.169	0.156	-0.524	0.103	0.096
	-0.500	-0.558	0.162	0.158	-0.525	0.100	0.095
	-0.500	-0.557	0.170	0.157	-0.528	0.095	0.095
	-0.500	-0.560	0.168	0.157	-0.524	0.101	0.096
	-0.500	-0.550	0.165	0.156	-0.524	0.094	0.095
	-0.500	-0.557	0.168	0.157	-0.523	0.096	0.095
	-0.500	-0.546	0.161	0.157	-0.525	0.098	0.096
	-0.500	-0.557	0.164	0.157	-0.527	0.097	0.096
	-0.500	-0.555	0.163	0.157	-0.521	0.097	0.095
	-0.500	-0.554	0.159	0.156	-0.526	0.102	0.095
β	1.000	0.972	0.226	0.237	0.995	0.174	0.177
	0.500	0.485	0.206	0.207	0.506	0.141	0.143
	1.000	0.962	0.220	0.238	0.999	0.170	0.178
	-1.000	-0.970	0.222	0.238	-1.008	0.171	0.179
	-1.000	-0.974	0.219	0.238	-0.999	0.172	0.178
	-1.000	-0.969	0.222	0.238	-0.996	0.172	0.177
	-1.000	-0.962	0.215	0.238	-1.008	0.169	0.178
	-1.000	-0.968	0.223	0.238	-0.996	0.169	0.178
	-1.000	-0.966	0.220	0.238	-1.012	0.171	0.179
	-1.000	-0.959	0.216	0.237	-1.000	0.172	0.178
SIM		0.950	(0.021)		0.981	(0.007)	

Table 2: Averages of classification accuracies (CA) of MV-logistic/Logistic regression and winning proportion (WP) of MV-logistic regression for different values of σ , p , q and $n = 150$. ρ_σ is the average of the explained proportion of the best rank-1 approximation.

(p, q)	(12, 10)		(20, 15)		(20, 30)	
σ	ρ_σ	CA (WP)	ρ_σ	CA (WP)	ρ_σ	CA (WP)
0	100%	0.867/0.739 (1.00)	100%	0.866/0.670 (1.00)	100%	0.839/0.622 (1.00)
0.1	69%	0.856/0.741 (1.00)	63%	0.848/0.668 (1.00)	58%	0.821/0.622 (1.00)
0.3	44%	0.794/0.745 (0.87)	36%	0.765/0.669 (0.94)	32%	0.720/0.623 (0.94)
0.5	33%	0.727/0.751 (0.37)	26%	0.677/0.673 (0.60)	22%	0.631/0.621 (0.61)

Table 3: The leave-one-out classification accuracy of MV-logistic/Logistic regression for the EEG Database Data Set under different combinations of (p_0, q_0) .

p_0	q_0		
	15	20	30
15	0.861 /0.803	0.836/0.795	0.844/0.787
30	0.844/0.779	0.828/0.779	0.828/0.754
60	0.844/0.737	0.853/0.713	0.828/0.713

Table 4: The leave-one-out classification accuracy of PCA followed by conventional logistic regression for the EEG Database Data Set under different choices of r .

r	1	2	3	4	5	6	7	8	9	10
Accuracy	0.631	0.648	0.730	0.820	0.820	0.803	0.787	0.779	0.787	0.795
r	20	30	40	50	60	70	80	90	100	120
Accuracy	0.746	0.779	0.795	0.754	0.738	0.746	0.713	0.705	0.631	0.492

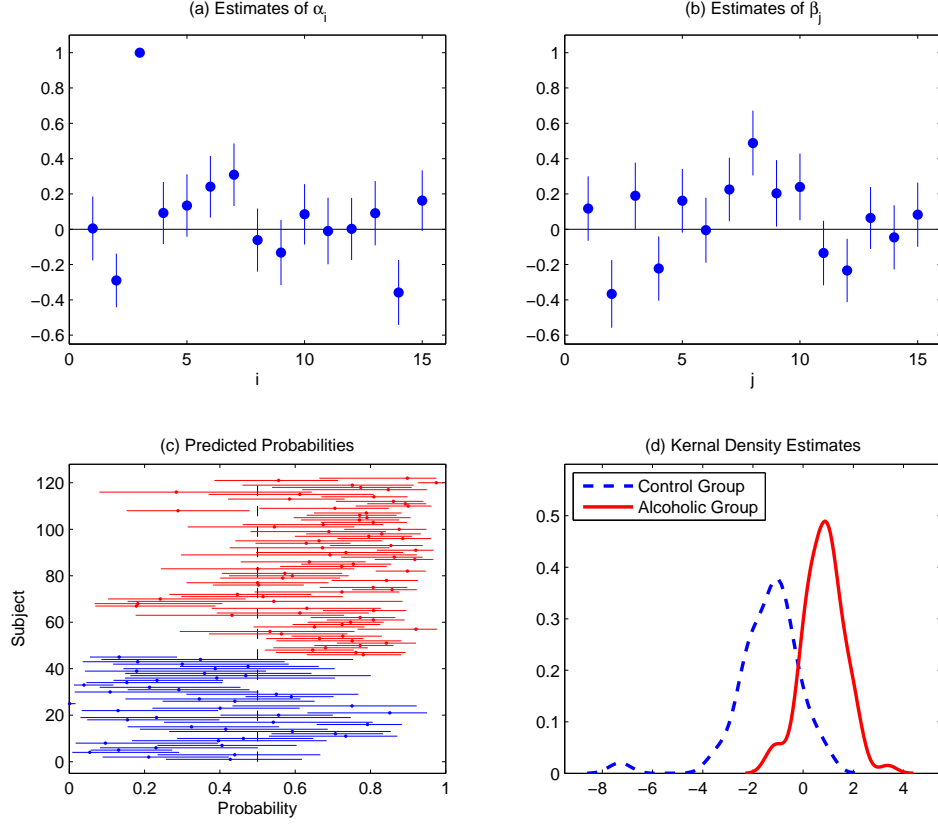


Figure 1: Analysis results of the EEG Database Data Set with $(p_0, q_0) = (15, 15)$. (a)-(b) Estimates of α and β (the circles) with the 95% confidence intervals (the vertical bars). As we set $\alpha_3 = 1$, no confidence interval is provided for α_3 . (c) Estimates of $\pi(\theta|X_i)$ (the symbol $*$) with 95% confidence intervals (the horizontal lines). Subjects 1–45 and 46–122 belong to the control and alcoholic groups, respectively. The vertical dash line indicates a probability of value 0.5. (d) Kernel density estimates of $\hat{\alpha}^T X_i \hat{\beta}$ for control and alcoholic groups.